Name:______________________________

Class:______________________________

GOSFORD HIGH SCHOOL

2014

Year 12 HSC COURSE

MATHEMATICS EXTENSION 1

ASSESSMENT TASK 2

General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 8 – 12, show relevant mathematical reasoning and/or calculations

Q1-7 Multiple Choice		7
Q8 Preliminary		8
Q8 Mathematical Induction		3
Q9 Further Graphs		7
Q10 Parabola		7
Q11 Iterative Methods		6
Q11 Methods of Integration		7
Q12 Binomial Expansion		7
TOTAL		52

Question 1 What is the Cartesian equation of the parabola with parametric form

$$x = 6t$$
$$y = -3t^2$$

A. $x^2 = -6y$ B. $x^2 = -12y$

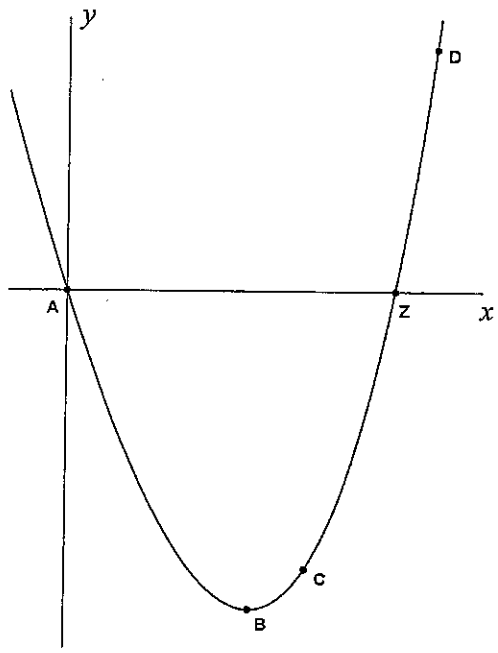
C. $x^2 = 6y$ D. $x^2 = 12y$

Question 2 The graph of $y = \frac{1}{x^2+1}$ has vertical asymptotes at:

A. $x = \pm 1$ B. $x = 1$

C. $x = 0$ D. It has none

Question 3 Which point is the most effective first estimate for the root at z when using Newton's method:



Question 4 Which of the following is an expression for $\int (2x+1)^{10}\, dx$?
Use the substitution $u = 2x + 1$.

A. $\frac{1}{9}(2x+1)^9 + C$ B. $\frac{1}{18}(2x+1)^9 + C$

C. $\frac{1}{11}(2x+1)^{11} + C$ D. $\frac{1}{22}(2x+1)^{11} + C$

Question 5 In the expansion of $(2x+b)^6$ the coefficients of x and x^2 are equal.
What is the value of b?

A. 5 B. 6

C. 11 D. 12

Question 6 The equation of the chord of contact to the parabola $x^2 = 4ay$ from the point external to the parabola (x_0, y_0)

A. $xx_0 = 2a(y + y_0)$ B. $xx_0 = 2a(y - y_0)$
C. $xx_0 = 4a(y + y_0)$ D. $xx_0 = 4a(y - y_0)$

Question 7 Given that $\frac{d}{dx}(f(x)) = 3x^2 f(x)$ find the value of

$$\int 3x^2 f(x)(f(x))^5 dx$$

A. $5(f(x))^4 + C$ B. $\frac{(f(x))^6}{6} + C$
C. $15(f(x))^4 + C$ D. $\frac{(f(x))^6}{2} + C$

Question 8 **START A NEW PAGE**

a) Solve 2

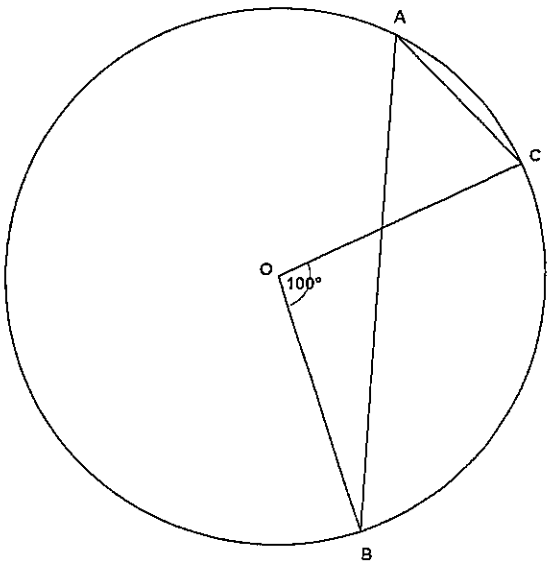
$$\frac{1}{(x-2)(x+1)} \leq 0$$

b) 4 women and 3 men stand in a line. How many arrangements are possible with:

i) No restrictions 1

ii) All the men together 1

c) Find the size of $\angle BAC$ giving reasons 2



d) Find the coordinates of the point P which divides the interval $(-2,4)$ to $(6,1)$ externally in the ratio $2:3$ 2

e) Prove by mathematical induction that for all $n \geq 10$ 3

$$2^n > 10n + 7$$

Question 9 START A NEW PAGE

a) For the function

$$f(x) = \frac{2+x}{1-x^2}$$

Find:

i) x and y intercepts **1**

ii) Any vertical asymptotes **1**

iii) Any horizontal asymptotes **1**

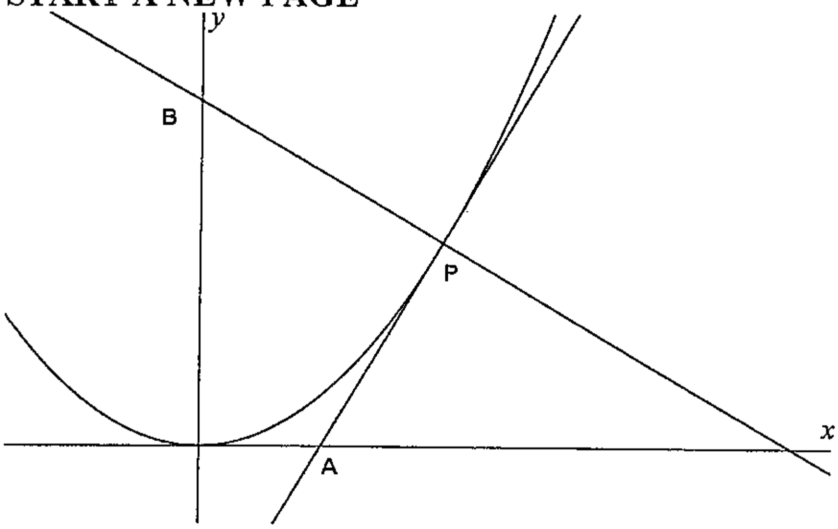
iv) Sketch the graph of $y = f(x)$ on $\frac{1}{3}$ of a page clearly showing all these features (it is not necessary to use calculus) **2**

b) What is the equation of the oblique asymptote of the equation **2**

$$y = \frac{x^3 + x^2 - 5x - 6}{x^2 + 1}$$

Question 10 START A NEW PAGE

a)



The diagram shows the graph of the parabola $x^2 = 4y$. The tangent to the parabola at $P(2p, p^2), p > 0$ cuts the x −axis at A. The normal to the parabola at P with equation $x + py = 2p + p^3$ cuts the ~~x~~ y −axis at B.

i) Derive the equation of the tangent AP **1**

ii) Show that B has coordinates $(0, p^2 + 2)$ **1**

iii) Let C be the midpoint of AB. Find the Cartesian equation of the locus of C. **2**

b) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

i) The tangent at P with equation $y = px - ap^2$ and the line through Q parallel to the y −axis intersect at T. **1**
Find the coordinates of T.

ii) Write down the coordinates of M, the midpoint of PT. **1**

iii) Determine the locus of M when $pq = -1$. **1**

Question 11 **START A NEW PAGE**

a) Consider the equation $2^{-x} = x$

i) By using the method of halving the interval show that there is a solution in the interval $0 \leq x \leq 4$ 2

ii) Find the solution correct to the nearest whole number. 2

b) Using 1 application of Newton's method and the first estimate $x_0 = 1$, find a better approximation for the solution of the equation $e^{2x} - 6 = e^x$ to 2 decimal places 2

c) Use the substitution $u = 3 - x^2$ to find 2

$$\int 2x\sqrt{3 - x^2}\, dx$$

d) Using the substitution $u = 2x + 3$ find 2

$$\int x\sqrt{2x + 3}\, dx$$

e) Evaluate 3

$$\int_3^6 \frac{x}{\sqrt{x - 2}}\, dx$$

Using the substitution $x = t^2 + 2$

Question 12 **START A NEW PAGE**

a) The 4th term in the expansion of $(ax + b)^n$ is ${}^8C_3 \times 2^3 \times 3^5 \times x^5$. 2
What are the values of a, b and n?

b) Find the value of the term that does not depend on x in the expansion of 2

$$\left(x^2 + \frac{3}{x}\right)^6$$

c) Consider the expansion of $(2x + 3)^{13}$

i) Show that the ratio of two consecutive coefficients C_{k+1} to C_k is given by 2

$$\frac{C_{k+1}}{C_k} = \frac{14 - k}{k} \times \frac{3}{2}$$

ii) Hence or otherwise, find the term with the greatest coefficient in the expansion of $(2x + 3)^{13}$ 1

HSC MATHEMATICS

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Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

1. A ○ B ○ C ● D ○
2. A ○ B ● C ○ D ○
3. A ○ B ○ C ● D ○
4. A ● B ○ C ○ D ○
5. A ○ B ○ C ○ D ●

Math Ext 1

Q1) B
2) D
3) D
4) D
5) ${}^6C_5\, 2^{6-5}\, b^5 = {}^6C_4\, 2^{6-4}\, b^4$

$12b^5 = 60b^4$

$b = 5$

A

6) A
7) B

3) a) consider denominator equal zero

$x = 2$ or $x = -1$

equality never occurs.

Test

x ✓ x

-1 2

$\therefore\ -1 < x < 2$

b) i) $7! = 5040$

ii) $5! \times 3! = 720$

c) $\angle BAC = 50^\circ$ (angle at circumference is half angle at the centre standing on same arc).

d) $(-2,4)$ $(6,1)$

$-2:3$

$$P = \left(\frac{-2\times3+6\times-2}{-2+3}, \frac{4\times3+1\times-2}{-2+3}\right)$$

$$= (-18, 10)$$

e) for $n=10$ $\quad LHS = 2^{10} = 1024$ $\quad RHS = 10\times10+7 = 107$

$\therefore$ LHS > RHS

$\therefore$ true for $n=10$

Assume true for $n=k$

$2^k > 10k+7$

Using this,

Prove true for $n=k+1$

ie RTP $2^{k+1} > 10(k+1)+7$

$$\begin{aligned} LHS &= 2^{k+1} \\ &= 2\times2^k \\ &> 2(10k+7) \quad \text{by assumption.} \\ &= 20k+14 \\ &= 10k+10+10k+4 \\ &= 10(k+1)+10k+4 \\ &> 10(k+1)+7 \quad \text{since } 10k+4>7 \text{ as } k\geq10 \\ &= RHS \end{aligned}$$

$\therefore$ true for $n=k+1$ if true for $n=k$.

$\therefore$ $2^n > 10n+7$ for all $n \geq 10$ by mathematical induction

One Alternative: (there were many others)

i.e $2^{k+1} > 10k+17$

$$\begin{aligned} LHS = 2^{k+1} &= 2\times2^k \\ &> 2(10k+7) \quad \text{(by assumption)} \\ &> 20k+14 \\ &> 19k+18 \quad \text{since } k\geq10 \\ &> 10k+17 \end{aligned}$$

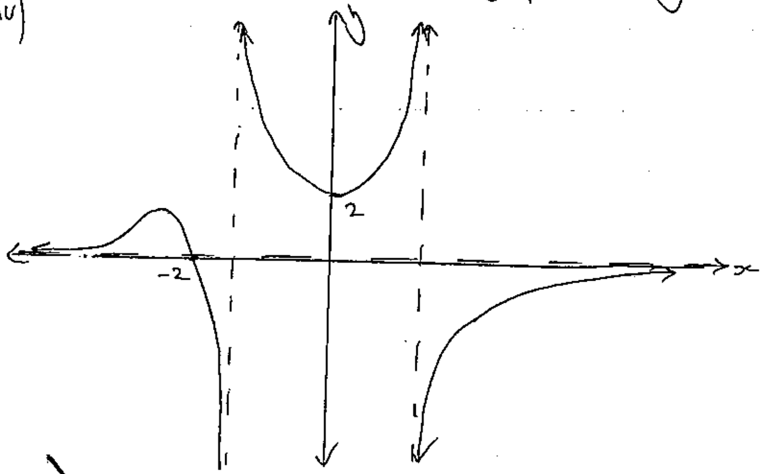
9) a) i) x-int: $x=-2$ $(-2,0)$ y-int: $y=2$ $(0,2)$

ii) $x=\pm 1$ are vertical asymptotes

iii) $\lim\limits_{x\to\infty} \frac{2+x}{1-x^2} = 0^-$ $\lim\limits_{x\to-\infty} \frac{2+x}{1-x^2} = 0^+$

$\therefore$ horizontal asymptote $y=0$

iv)



b)

$$\begin{array}{r} x+1 \\ x^2+0x+1 \overline{)\, x^3+x^2-5x-6} \\ x^3+0x^2+x \\ \hline x^2-6x-6 \\ x^2+0x+1 \\ \hline -6x-7 \end{array}$$

$\therefore$ $y=x+1$ is the asymptote

10) a) i) $y=\frac{x^2}{4}$

$y'=\frac{x}{2}$

@ $x=2p$ $m=p$

$\therefore$ $y-p^2 = p(x-2p)$

$y-p^2 = px-2p^2$

$y = px-p^2$

ii) B is y-int of normal

$0+py = 2p+p^3$

$py = p(p^2+2)$

$y = p^2+2$

$\therefore B = (0, p^2+2)$

iii) $y=px-p^2$

$0=px-p^2$

$x=p$

$A=(p,0)$

$C = \left(\frac{p+0}{2}, \frac{p^2+2+0}{2}\right)$

$p=2x$ $\therefore x>0$

$y=\frac{(2x)^2+2}{2}$

$y=\frac{4x^2+2}{2}$

$=2x^2+1$ for $x>0$

b) i) through Q $\parallel$ to y-axis

$x=2aq$

$y=px-ap^2$

$y=2apq-ap^2$

$T=(2aq, 2apq-ap^2)$

ii) $M=\left(\frac{2ap+2aq}{2}, \frac{ap^2+2apq-ap^2}{2}\right)$

$=\left(a(p+q), apq\right)$

iii) $y=-a$ is the locus

Q 11) a) i) $f(x) = 2^{-x} - x$

$f(0) = 1$

$f(4) = -3\frac{15}{16}$

since sign changes there must be a root on $0 \le x \le 4$

ii) $f(2) = -1\frac{3}{4}$

$\therefore$ root between 0 & 2 since sign changes

$f(1) = -\frac{1}{2}$

$\therefore$ root between 0 & 1 since sign changes

$f(0.5) = 0.207\ldots$

$\therefore$ root between 0.5 & 1 since sign changes.

$\therefore$ solution is 1 correct to nearest whole number.

b) $f(x) = e^{2x} - e^{x} - 6$ $\qquad f(1) = -1.329\ldots$

$f'(x) = 2e^{2x} - e^{x}$ $\qquad f'(1) = 12.059\ldots$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 1.1102\ldots$

$= 1.11$

c) $\int 2x\sqrt{3-x^2}\,dx = -\int \sqrt{u}\,du$ $\qquad u = 3 - x^2$, $\frac{du}{dx} = -2x$

$= -\int u^{\frac{1}{2}}\,du$

$= -\frac{2}{3}u^{\frac{3}{2}} + C$

$= \dfrac{-2\left(\sqrt{3-x^2}\right)^3}{3} + C$

d) $\int x\sqrt{2x+3}\,dx = \frac{1}{2}\int \frac{u-3}{2}\sqrt{u}\,du$ $\qquad u = 2x+3$, $\frac{du}{dx} = 2$

$= \frac{1}{4}\int u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\,du$

$= \frac{1}{4}\left(\frac{2u^{\frac{5}{2}}}{5} - 2u^{\frac{3}{2}}\right) + C$

$= \dfrac{(2x+3)^{\frac{5}{2}}}{10} - \dfrac{(2x+3)^{\frac{3}{2}}}{2} + C$

e) $\int_3^6 \dfrac{x}{\sqrt{x-2}}\,dx = \int_1^2 \dfrac{t^2+2}{\sqrt{t^2+2-2}} \times 2t\,dt$ $\qquad x = t^2+2$, $\frac{dx}{dt} = 2t$

when $x = 3$ $\quad t = 1$

$x = 6$ $\quad t = 2$

$= 2\int_1^2 t^2 + 2\,dt$

$= 2\left[\frac{t^3}{3} + 2t\right]_1^2$

$= 2\left(\frac{8}{3} + 4 - \left(\frac{1}{3} + 2\right)\right)$

$= \frac{26}{3}$

12) a) $T_{k+1} = {}^nC_k\,(ax)^{n-k}\,b^k$

$k = 3$

$n = 8 \qquad a = 3 \qquad b = 2$

b) $T_{k+1} = {}^6C_k\,(x^2)^{6-k}\left(\frac{3}{x}\right)^k$

$= {}^6C_k\,3^k \times x^{12-2k} \times x^{-k}$

$= {}^6C_k \times 3^k \times x^{12-3k}$

independent of x: $k = 4$

$\therefore$ term ${}^6C_4 \times 3^4 = 1215$

c) i) $T_{k+1} = {}^{13}C_k \times (2x)^{13-k}\,3^k$

$\therefore C_{k+1} = {}^{13}C_k \times 2^{13-k} \times 3^k$

$T_k = {}^{13}C_{k-1} \times (2x)^{13-(k-1)} \times 3^{k-1}$

$\therefore C_k = {}^{13}C_{k-1} \times 2^{14-k} \times 3^{k-1}$

$\therefore \dfrac{C_{k+1}}{C_k} = \dfrac{{}^{13}C_k \times 2^{13-k} \times 3^k}{{}^{13}C_{k-1} \times 2^{14-k} \times 3^{k-1}}$

$= \dfrac{13!}{(13-k)!\,k!} \div \dfrac{13!}{(13-(k-1))!\,(k-1)!} \times 2^{-1} \times 3^{1}$

$= \dfrac{13!}{(13-k)!\,k(k-1)\cancel{k}!} \times \dfrac{(14-k)(13-k)!\,(k-1)!}{13!} \times \dfrac{3}{2}$

$= \dfrac{14-k}{k} \times \dfrac{3}{2}$

ii) consider $\frac{14-k}{k} \times \frac{3}{2} > 1$

$$42 - 3k > 2k$$
$$5k < 42$$
$$k < 8.4$$

$\therefore$ greatest coefficient when $k = 8$

$\therefore$ ${}^{13}C_8\, 2^5 3^8$ is greatest coefficient

$(= 270\ 208\ 224)$